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k-Product Cordial Labeling of Splitting Graph of Star Graphs

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Abstract

Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leqslant k \leqslant |V(G)|$. For each edge uv assign the label $f(u)f(v)(\text{mod } k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leqslant 1$, and $|e_f(i) - e_f(j)| \leqslant 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. A graph that admits k-product cordial labeling is called k-product cordial graph. We have already proved that several families of graphs admit k-product cordial labeling. In this paper, we show that the splitting graph of star graphs admit k-product cordial labeling.

Keywords: cordial labeling, product cordial labeling, k-product cordial labeling, splitting graph, star graph.

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1. Introduction and Terminology

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3]. The concept of labeling of graph has gained a lot of popularity in the field of graph theory. In 1967, Rosa [14] published a pioneering paper on graph labeling problems. Since then, many graph labeling techniques have been introduced and studied by several authors (refer [2]). Cordial labeling is one of the popular labelings defined by Cahit [1] in 1987: Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Followed by this, in 2004 Sundaram et al. [15] introduced a variation of cordial labeling called product cordial labeling: Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i (i = 0, 1)$. Several papers have been published on this topic.

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Later in 2012, Ponraj et al. [13] further extended the concept of product cordial labeling and defined k-product cordial labeling as follows: Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leqslant k \leqslant |V(G)|$. For each edge uv assign the label $f(u)f(v)(\mathrm{mod}\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leqslant 1$, and $|e_f(i) - e_f(j)| \leqslant 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. A graph that admits k-product cordial labeling is called k-product cordial graph. We have already proved that several families of graphs admit k-product cordial labeling. The concept of k-product cordial labeling and the work of the authors in [13], motivated us to do further research on this topic. Consequently in our research, we established that the following families of graphs admit k-product cordial labeling: union of graphs [4]; cone and double cone graphs [5]; Napier bridge graphs [6]; product of graphs [7]; powers of paths [8]; fan and double fan graphs [9]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil \frac{p-1}{4} \rceil \lfloor \frac{p-1}{4} \rfloor + 3$ [10] and path graphs [11]. Jeyanthi et al. [12] already proved that the splitting graph $S'(K_{1,n})$ is a 3-product cordial graph. In this paper, we show that splitting graph of star graphs admit k-product cordial labeling for $k \geqslant 4$.

We use the following terminology to prove our main results. The splitting graph $S'(G)$ of a graph G is a graph obtained by adding a new vertex v' to each vertex v of G such that v' is adjacent to every vertex that is adjacent to v in G. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. If every vertex of V_1 is adjacent with every vertex of V_2, then G is a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a star graph. An illustration of $S'(K_{1,4})$ is shown in Figure 1.

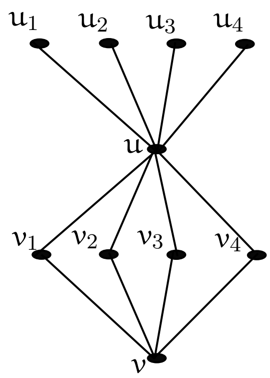

Figure 1: Splitting graph of star graph $K_{1,4}$

2. Main Results

Theorem 2.1. The graph $S'(K_{1,n})$ is k-product cordial for $n \geqslant \frac{k}{2}$ where k is even and $k \geqslant 4$.

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i\ ;\ 1 \leqslant i \leqslant n\}$ and $E(S'(K_{1,n})) = \{(u, u_i)\ ;\ 1 \leqslant i \leqslant n\} \cup \{(v, v_i)\ ;\ 1 \leqslant i \leqslant n\} \cup \{(u, v_i)\ ;\ 1 \leqslant i \leqslant n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \to \{0, 1, 2, ..., k-1\}$ for $k \geqslant 4$ as follows:

Case (i): If $n \equiv 0(\mathrm{mod}\ k)$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\mathrm{mod}\ k)$ and $f(v_i) = i(\mathrm{mod}\ k)$ for $1 \leqslant i \leqslant n$.

Case (ii): If $n \equiv k-1(\mathrm{mod}\ k)$, then

$f(u) = 1$, $f(u_{n-1}) = 0$, $f(u_n) = 2$, $f(v) = k-1$, $f(v_{n-1}) = 0$, $f(v_n) = 1$, $f(u_i) = i(\mathrm{mod}\ k)$ and $f(v_i) = i(\mathrm{mod}\ k)$ for $1 \leqslant i \leqslant \lfloor \frac{n}{k} \rfloor k$.

For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leqslant j \leqslant k-3$,

$f(u_i) = j+1$ and $f(v_i) = j+2$.

Case (iii): If $n \equiv x(\mathrm{mod}\ k)$; $1 \leqslant x \leqslant \frac{k}{2} - 1$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leqslant i \leqslant k\left(\lfloor \frac{n}{k} \rfloor - 1\right)$.
For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j$; $1 \leqslant j \leqslant 2x$,
$$f(u_i) = \begin{cases} \frac{k}{2} & \text{if } j = k-3 \\ 0 & \text{if } j = k-2. \end{cases}$$
$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1-\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$
For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x + j$; $1 \leqslant j \leqslant n - (k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x)$,
$$f(u_i) = \begin{cases} 0 & \text{if } j = k-2 \\ 1 & \text{if } j = k-1. \end{cases}$$
$$f(u_i) = \begin{cases} k-1-\frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1 + \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$
For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j$; $1 \leqslant j \leqslant k-3$, $f(v_i) = j+1$.
For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j$; $1 \leqslant j \leqslant n - (k\lfloor \frac{n}{k} \rfloor - 3)$,
$$f(v_i) = \begin{cases} k-1 & \text{if } j = 1,4 \\ 1 & \text{if } j = 2,5 \\ 0 & \text{if } j = 3,6 \\ \frac{k}{2} & \text{if } j = 7. \end{cases}$$
If $j \geqslant 8$, then $f(v_i) = \begin{cases} \frac{k}{2} - \frac{j-6}{2} & \text{if } j \text{ is even} \\ \frac{k}{2} + \frac{j-7}{2} & \text{if } j \text{ is odd.} \end{cases}$

Case (iv): If $n \equiv k - x(\text{mod } k)$; $2 \leqslant x \leqslant \frac{k}{2}$, then
$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leqslant i \leqslant k\lfloor \frac{n}{k} \rfloor$.
For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leqslant j \leqslant k - 2x$,
$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1-\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$
For $i = k\lfloor \frac{n}{k} \rfloor + k - 2x + j$; $1 \leqslant j \leqslant x$,
$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1-\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$
For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leqslant j \leqslant 4$,
$$f(v_i) = \begin{cases} 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \\ k-1 & \text{if } j = 3 \\ \frac{k}{2} & \text{if } j = 4. \end{cases}$$
For $i = k\lfloor \frac{n}{k} \rfloor + 4 + j$; $1 \leqslant j \leqslant n - (k\lfloor \frac{n}{k} \rfloor + 4)$,
$$f(v_i) = \begin{cases} \frac{k}{2} + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ \frac{k}{2} - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$
From the above cases, we have
for $n \equiv x(\text{mod } k)$; $x = \frac{k}{2} - 1$ or $k-1$, $v_f(i) = \frac{2n+2}{k}$ for $0 \leqslant i \leqslant k-1$.
If $n \equiv x(\text{mod } k)$; $0 \leqslant x \leqslant \frac{k}{2} - 2$, then
$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leqslant i \leqslant x+1,\ k-(x+1) \leqslant i \leqslant k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x+2 \leqslant i \leqslant k-(x+2). \end{cases}$$
If $n \equiv (\frac{k}{2} + x)(\text{mod } k)$; $0 \leqslant x \leqslant \frac{k}{2} - 2$ and $k > 4$, then
$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leqslant i \leqslant x+1,\ k-(x+1) \leqslant i \leqslant k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x+2 \leqslant i \leqslant k-(x+2). \end{cases}$$
If $k = 4$ and $n \equiv 2(\text{mod } 4)$, then
$$v_f(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 1,2 \\ 2\lfloor \frac{n}{k} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Case 1: If $n \equiv 0 (\bmod\ k)$, then $e_f(i) = \frac{3n}{k}$ for $0 \leqslant i \leqslant k-1$.
Case 2: If $n \equiv k-1 (\bmod\ k)$, then

$$e_f(i) = \begin{cases} 3\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 1, k-1, k-2 \\ 3\lfloor \frac{n}{k} \rfloor + 3 & \text{otherwise.} \end{cases}$$

Case 3: If $n \equiv x (\bmod\ k)$; $1 \leqslant x \leqslant \lfloor \frac{k}{3} \rfloor$; $k > 4$, then for $x = 1$

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, k-2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $k = 4$ and $x = 1$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $k = 6$ and $x = 2$,

$e_f(i) = 3(\lfloor \frac{n}{k} \rfloor - 1) + 4$; $0 \leqslant i \leqslant 5$,

for $k > 6$ and $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, 3, k-1, k-2, k-3 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 3$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2, 3, 4, k-1, k-2, k-3, k-4 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leqslant x \leqslant \lfloor \frac{k}{3} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \ldots, x+1, k-(x+1), \\ & \quad \frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, \ldots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leqslant x \leqslant \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \ldots, x+1, k-(x+1), \\ & \quad \frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, \ldots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv x (\bmod\ k)$ where $k \equiv 0 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 1 \leqslant x \leqslant \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \ldots, x+1, k-(x+1), \\ & \quad \frac{k}{2} + (\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{6} \rfloor - 1), \ldots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leqslant x \leqslant \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \ldots, x+1, k-(x+1), \\ & \quad \frac{k}{2} + (\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{6} \rfloor - 1), \ldots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x (\bmod\ k)$ where $k \equiv 0 (\bmod\ 3)$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \ldots, x, k-x, x+1, 0, \\ & \quad \frac{k}{2} + (\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{6} \rfloor - 1), \ldots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \ldots, x, k-x, x+1, 0, \\ & \quad \frac{k}{2} + (\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{6} \rfloor - 1), \ldots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x (\bmod\ k)$ where $k \equiv 1 (\bmod\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = x+1, k-(x+1), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv x (\bmod\ k)$ where $k \equiv 1 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k-(\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k-(\lfloor \frac{k}{3} \rfloor + 3), \ldots, x+1, k-(x+1), \\ & \quad \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \ldots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2} - 2$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+2, k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x+1, k-(x+1),\\ & \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 5: If $n\equiv x(\bmod k)$ where $k\equiv 1(\bmod 3)$, $x=\frac{k}{2}-1$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+2, k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x, k-x, x+1, 0,\\ & \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}-(\lfloor\frac{x}{2}\rfloor-2), \frac{k}{2}+(\lfloor\frac{x}{2}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

for $x=\frac{k}{2}-1$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+2, k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x, k-x, x+1, 0,\\ & \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 6: If $n\equiv x(\bmod k)$ where $k\equiv 2(\bmod 3)$, $x=\lfloor\frac{k}{3}\rfloor+1$,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=k-(\lfloor\frac{k}{3}\rfloor+2),\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 7: If $n\equiv x(\bmod k)$ where $k\equiv 2(\bmod 3)$, $\lfloor\frac{k}{3}\rfloor+2\leqslant x\leqslant\frac{k}{2}-2$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x+1, k-(x+1),\\ & \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}-(\lfloor\frac{x}{2}\rfloor-2), \frac{k}{2}+(\lfloor\frac{x}{2}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+2\leqslant x\leqslant\frac{k}{2}-2$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x+1, k-(x+1),\\ & \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 8: If $n\equiv x(\bmod k)$ where $k\equiv 2(\bmod 3)$, $x=\frac{k}{2}-1$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x, k-x, x+1, 0,\\ & \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}-(\lfloor\frac{x}{2}\rfloor-2), \frac{k}{2}+(\lfloor\frac{x}{2}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

for $x=\frac{k}{2}-1$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=k-(\lfloor\frac{k}{3}\rfloor+2), \lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), ..., x, k-x, x+1, 0,\\ & \frac{k}{2}-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \frac{k}{2}+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 9: If $k=8$ and $n\equiv 3(\bmod 8)$, then

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=0\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Case 4: If $n\equiv k-x(\bmod k)$; $2\leqslant x\leqslant\lfloor\frac{k}{3}\rfloor$ and x is odd, then

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+2 & \text{if } i=\frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, ..., \frac{k}{2}+(x-2), \frac{k}{2}-(x-2),\\ & 0, k-1, 1, k-2, 2, ..., (\lfloor\frac{x}{2}\rfloor+1), k-(\lfloor\frac{x}{2}\rfloor+2)\\ 3(\lfloor\frac{n}{k}\rfloor)+3 & \text{otherwise.}\end{cases}$$

for $2\leqslant x\leqslant\lfloor\frac{k}{3}\rfloor$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+2 & \text{if } i=\frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, ..., \frac{k}{2}+(x-2), \frac{k}{2}-(x-2),\\ & 0, k-1, 1, k-2, 2, ..., k-(\frac{x}{2}+1), \frac{x}{2}+1\\ 3(\lfloor\frac{n}{k}\rfloor)+3 & \text{otherwise.}\end{cases}$$

Subcase 1: If $n\equiv k-x(\bmod k)$ where $k\equiv 0(\bmod 3)$, $\lfloor\frac{k}{3}\rfloor+1\leqslant x\leqslant\frac{k}{2}$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\frac{k}{2}+(\frac{k}{3}-1), \frac{k}{2}-(\frac{k}{3}-1), \frac{k}{2}+\frac{k}{3}, \frac{k}{2}-\frac{k}{3}, ..., \frac{k}{2}+(x-2), \frac{k}{2}-(x-2),\\ & k-(\lfloor\frac{k}{6}\rfloor+2), (\lfloor\frac{k}{6}\rfloor+2), ..., (\lfloor\frac{x}{2}\rfloor+1), k-(\lfloor\frac{x}{2}\rfloor+2)\\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+1\leqslant x\leqslant\frac{k}{2}$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\frac{k}{2}+(\frac{k}{3}-1), \frac{k}{2}-(\frac{k}{3}-1), \frac{k}{2}+\frac{k}{3}, \frac{k}{2}-\frac{k}{3}, ..., \frac{k}{2}+(x-2), \frac{k}{2}-(x-2),\\ & k-(\lfloor\frac{k}{6}\rfloor+2), (\lfloor\frac{k}{6}\rfloor+2), ..., k-(\frac{x}{2}+1), (\frac{x}{2}+1)\\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 2: If $n\equiv k-x(\bmod k)$ where $k\equiv 1(\bmod 3)$ and $k>4$, $x=\lfloor\frac{k}{3}\rfloor+1$,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\frac{k}{2}+(\lfloor\frac{k}{3}\rfloor-1), \frac{k}{2}-(\lfloor\frac{k}{3}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 3: If $k = 4$, $n \equiv 2 (\mathrm{mod}\ 4)$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = 1, 3 \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv k - x (\mathrm{mod}\ k)$ where $k \equiv 1 (\mathrm{mod}\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - \lfloor \frac{k}{3} \rfloor, \\ & \ldots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \ldots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), \\ & \ldots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \ldots, k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv k - x (\mathrm{mod}\ k)$ where $k \equiv 2 (\mathrm{mod}\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv k - x (\mathrm{mod}\ k)$ where $k \equiv 2 (\mathrm{mod}\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \ldots, \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \\ & \ldots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \ldots, \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \ldots, k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ for all $0 \leqslant i, j \leqslant k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial for $n \geqslant \frac{k}{2}$ where k is even and $k \geqslant 4$. □

An example of 8-product cordial labeling of $S'(K_{1,10})$ shown in Figure 2.

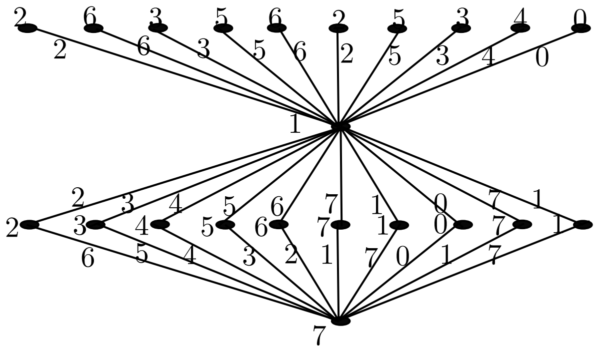

Figure 2: 8-Product cordial labeling of $S'(K_{1,10})$

Theorem 2.2. The graph $S'(K_{1,n})$ is k-product cordial for $n \geqslant \lfloor \frac{k}{2} \rfloor$ where k is odd and $k \geqslant 4$.

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i \ ; \ 1 \leqslant i \leqslant n\}$ and $E(S'(K_{1,n})) = \{(u, u_i) \ ; \ 1 \leqslant i \leqslant n\} \cup \{(v, v_i) \ ; \ 1 \leqslant i \leqslant n\} \cup \{(u, v_i) \ ; \ 1 \leqslant i \leqslant n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \to \{0, 1, 2, \ldots, k-1\}$ for $k \geqslant 4$ as follows:

Case (i): If $n \equiv 0 (\mathrm{mod}\ k)$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i (\mathrm{mod}\ k)$ and $f(v_i) = i (\mathrm{mod}\ k)$ for $1 \leqslant i \leqslant n$.

Case (ii): If $n \equiv k-1 (\mathrm{mod}\ k)$, then

$f(u) = 1$, $f(u_{n-1}) = 0$, $f(u_n) = 0$, $f(v) = k-1$.

$f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leqslant i \leqslant \lfloor \frac{n}{k} \rfloor k$.
For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leqslant j \leqslant k-3$, $f(u_i) = j+1$.
For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leqslant j \leqslant k-1$, $f(v_i) = j$.
Case (iii): If $n \equiv x(\text{mod } k)$; $1 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$, then
$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leqslant i \leqslant k\left(\lfloor \frac{n}{k} \rfloor - 1\right)$.
For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j$; $1 \leqslant j \leqslant 2x$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor - 1 \\ 1 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor. \end{cases}$$

$$f(u_i) = \begin{cases} k-1-\frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1+\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x + j$; $1 \leqslant j \leqslant n - (k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x)$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = k-2 \\ 1 & \text{if } j = k-1. \end{cases}$$

$$f(u_i) = \begin{cases} k-1-\frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1+\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j$; $1 \leqslant j \leqslant k-3$, $f(v_i) = 1+j$.
For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j$; $1 \leqslant j \leqslant n - (k\lfloor \frac{n}{k} \rfloor - 3)$,

$$f(v_i) = \begin{cases} k-1 & \text{if } j = 1,4 \\ 1 & \text{if } j = 2,5 \\ 0 & \text{if } j = 3,6. \end{cases}$$

If $j \geqslant 7$, then $f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1 - \frac{j-5}{2} & \text{if } j \text{ is odd} \\ \lfloor \frac{k}{2} \rfloor + \frac{j-6}{2} & \text{if } j \text{ is even.} \end{cases}$

Case (iv): If $n \equiv k-x(\text{mod } k)$; $2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$, then
$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leqslant i \leqslant k\lfloor \frac{n}{k} \rfloor$.
For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leqslant j \leqslant 2$, $f(u_i) = 0$.
For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leqslant j \leqslant 2(\lfloor \frac{k}{2} \rfloor - x)$,

$$f(u_i) = \begin{cases} 1+\frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1-\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x) + j$; $1 \leqslant j \leqslant n - (k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x))$,

$$f(u_i) = \begin{cases} 1+\frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1-\frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leqslant j \leqslant 2$, $f(v_i) = \begin{cases} 1 & \text{if } j = 1 \\ k-1 & \text{if } j = 2. \end{cases}$.
For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leqslant j \leqslant n - (k\lfloor \frac{n}{k} \rfloor + 2)$,

$$f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ \lfloor \frac{k}{2} \rfloor + 1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

From the above cases, we have
for $n \equiv \lfloor \frac{k}{2} \rfloor(\text{mod } k)$,

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } i = 1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } 2 \leqslant i \leqslant k-1. \end{cases}$$

If $n \equiv x(\text{mod } k)$; $0 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor - 1$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leqslant i \leqslant x+1,\ k-(x+1) \leqslant i \leqslant k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x+2 \leqslant i \leqslant k-(x+2). \end{cases}$$

If $n \equiv k-1(\text{mod } k)$, then $v_f(i) = \frac{2n+2}{k}$ for $0 \leqslant i \leqslant k-1$.
If $n \equiv (\lfloor \frac{k}{2} \rfloor + x)(\text{mod } k)$; $1 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor - 1$, then

$$\nu_f(i) = \begin{cases} \lfloor\frac{2n+2}{k}\rfloor + 1 & \text{if } 0 \leqslant i \leqslant x,\ k-x \leqslant i \leqslant k-1 \\ \lfloor\frac{2n+2}{k}\rfloor & \text{if } x+1 \leqslant i \leqslant k-(x+1). \end{cases}$$

Case 1: If $n \equiv 0(\text{mod } k)$, then $e_f(i) = \frac{3n}{k}$ for $0 \leqslant i \leqslant k-1$.

Case 2: If $n \equiv k-1(\text{mod } k)$, then

$$e_f(i) = \begin{cases} 3\lfloor\frac{n}{k}\rfloor + 2 & \text{if } i = 0, 1, k-1 \\ 3\lfloor\frac{n}{k}\rfloor + 3 & \text{otherwise.} \end{cases}$$

Case 3: If $n \equiv x(\text{mod } k)$; $1 \leqslant x \leqslant \lfloor\frac{k}{3}\rfloor$, then for $x = 1$

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{if } i = 1, 2, k-2 \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{if } i = 1, 2, 3, k-1, k-2, k-3 \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 3$,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{if } i = 0, 1, 2, 3, 4, k-1, k-2, k-3, k-4 \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leqslant x \leqslant \lfloor\frac{k}{3}\rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, ..., x+1, k-(x+1), \\ & \lceil\frac{k}{2}\rceil, \lceil\frac{k}{2}\rceil - 1, \lceil\frac{k}{2}\rceil + 1, ..., \lceil\frac{k}{2}\rceil + (\lfloor\frac{x}{2}\rfloor - 2), \lceil\frac{k}{2}\rceil - (\lfloor\frac{x}{2}\rfloor - 1) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leqslant x \leqslant \lfloor\frac{k}{3}\rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, ..., x+1, k-(x+1), \\ & \lceil\frac{k}{2}\rceil, \lceil\frac{k}{2}\rceil - 1, \lceil\frac{k}{2}\rceil + 1, ..., \lceil\frac{k}{2}\rceil - (\frac{x}{2} - 2), \lceil\frac{k}{2}\rceil + (\frac{x}{2} - 2) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv x(\text{mod } k)$ where $k \equiv 0(\text{mod } 3)$, $\lfloor\frac{k}{3}\rfloor + 1 \leqslant x \leqslant \lfloor\frac{k}{2}\rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x+1, k-(x+1), \\ & \lceil\frac{k}{2}\rceil + (\lfloor\frac{k}{6}\rfloor - 1), \lceil\frac{k}{2}\rceil - \lfloor\frac{k}{6}\rfloor, \lceil\frac{k}{2}\rceil + \lfloor\frac{k}{6}\rfloor, \\ & ..., \lceil\frac{k}{2}\rceil + (\lfloor\frac{x}{2}\rfloor - 2), \lceil\frac{k}{2}\rceil - (\lfloor\frac{x}{2}\rfloor - 1) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor\frac{k}{3}\rfloor + 1 \leqslant x \leqslant \lfloor\frac{k}{2}\rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x+1, k-(x+1), \\ & \lceil\frac{k}{2}\rceil + (\lfloor\frac{k}{6}\rfloor - 1), \lceil\frac{k}{2}\rceil - \lfloor\frac{k}{6}\rfloor, \lceil\frac{k}{2}\rceil + \lfloor\frac{k}{6}\rfloor, \\ & ..., \lceil\frac{k}{2}\rceil - (\frac{x}{2} - 2), \lceil\frac{k}{2}\rceil + (\frac{x}{2} - 2) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x(\text{mod } k)$ where $k \equiv 0(\text{mod } 3)$, $x = \lfloor\frac{k}{2}\rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, 0, 1, \\ & \lceil\frac{k}{2}\rceil + (\lfloor\frac{k}{6}\rfloor - 1), \lceil\frac{k}{2}\rceil - \lfloor\frac{k}{6}\rfloor, \lceil\frac{k}{2}\rceil + \lfloor\frac{k}{6}\rfloor, \\ & ..., \lceil\frac{k}{2}\rceil + (\lfloor\frac{x}{2}\rfloor - 2), \lceil\frac{k}{2}\rceil - (\lfloor\frac{x}{2}\rfloor - 1) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor\frac{k}{2}\rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, 0, 1, \\ & \lceil\frac{k}{2}\rceil + (\lfloor\frac{k}{6}\rfloor - 1), \lceil\frac{k}{2}\rceil - \lfloor\frac{k}{6}\rfloor, \lceil\frac{k}{2}\rceil + \lfloor\frac{k}{6}\rfloor, \\ & ..., \lceil\frac{k}{2}\rceil - (\frac{x}{2} - 2), \lceil\frac{k}{2}\rceil + (\frac{x}{2} - 2) \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x(\text{mod } k)$ where $k \equiv 1(\text{mod } 3)$ and $k > 7$, $x = \lfloor\frac{k}{3}\rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = \lfloor\frac{k}{3}\rfloor + 2, k-(\lfloor\frac{k}{3}\rfloor + 2), \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv 3(\text{mod } k)$ and $k = 7$,

$$e_f(i) = \begin{cases} 3(\lfloor\frac{n}{k}\rfloor - 1) + 5 & \text{if } i = 0, 1 \\ 3(\lfloor\frac{n}{k}\rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv x(\text{mod } k)$ where $k \equiv 1(\text{mod } 3)$, $\lfloor\frac{k}{3}\rfloor + 2 \leqslant x \leqslant \lfloor\frac{k}{2}\rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv x (\bmod\ k)$ where $k \equiv 1 (\bmod\ 3)$, $x = \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 7: If $n \equiv x (\bmod\ k)$ where $k \equiv 2 (\bmod\ 3)$ and $k > 5$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 8: If $n \equiv 2 (\bmod\ k)$ where $k = 5$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 1, \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 9: If $n \equiv x (\bmod\ k)$ where $k \equiv 2 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \lfloor \frac{k}{3} \rfloor + 4, k - (\lfloor \frac{k}{3} \rfloor + 4), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \lfloor \frac{k}{3} \rfloor + 4, k - (\lfloor \frac{k}{3} \rfloor + 4), \\ & ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 10: If $n \equiv x (\bmod\ k)$ where $k \equiv 2 (\bmod\ 3)$, $x = \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \lfloor \frac{k}{3} \rfloor + 4, k - (\lfloor \frac{k}{3} \rfloor + 4), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \lfloor \frac{k}{3} \rfloor + 4, k - (\lfloor \frac{k}{3} \rfloor + 4), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Case 4: If $n \equiv k - x (\bmod\ k)$; $2 \leqslant x \leqslant \lfloor \frac{k}{3} \rfloor$ and x is odd, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & 0, k-1, 1, k-2, 2, ..., k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

for $2 \leqslant x \leqslant \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & 0, k-1, 1, k-2, 2, ..., \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv k - x (\bmod\ k)$ where $k \equiv 0 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 1 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\frac{k}{3} - 1), \lfloor \frac{k}{2} \rfloor + \frac{k}{3}, \lfloor \frac{k}{2} \rfloor - \frac{k}{3}, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), ..., k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\frac{k}{3} - 1), \lfloor \frac{k}{2} \rfloor + \frac{k}{3}, \lfloor \frac{k}{2} \rfloor - \frac{k}{3}, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), ..., \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv k - x (\bmod\ k)$ where $k \equiv 1 (\bmod\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv k - x (\bmod\ k)$ where $k \equiv 1 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), ..., k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), ..., \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv k - x (\bmod\ k)$ where $k \equiv 2 (\bmod\ 3)$ and $k > 5$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv k - x (\bmod\ k)$ where $k = 5$, $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = 2, 3 \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv k - x (\bmod\ k)$ where $k \equiv 2 (\bmod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor + 1), \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor + 1), \\ & ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2, ..., k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leqslant x \leqslant \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor + 1), \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor + 1), \\ & ..., \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), ..., \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ for all $0 \leqslant i, j \leqslant k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial for $n \geqslant \lfloor \frac{k}{2} \rfloor$ where k is odd and $k \geqslant 4$. □

An example of 15-product cordial labeling of $S'(K_{1,11})$ is shown in Figure 3.

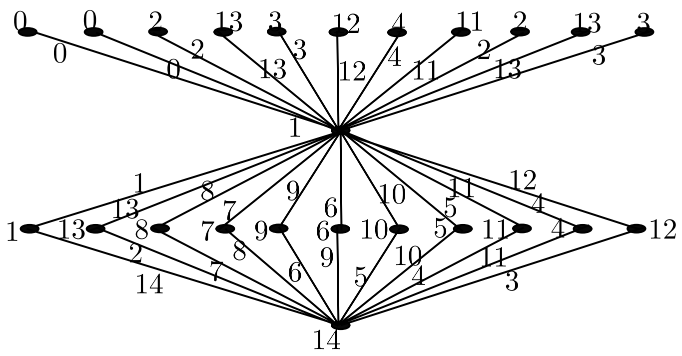

Figure 3: 15-product cordial labeling of $S'(K_{1,11})$

Theorem 2.3. The graph $S'(K_{1,n})$ is k-product cordial if and only if $n \geqslant 1$ and $k \geqslant 4$ except $\lfloor \frac{k}{3} \rfloor \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 1, 0 \pmod 3$ for $k = 4, 6$ and $k \geqslant 8$ and $\lceil \frac{k}{3} \rceil \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 2 \pmod 3$ for $k \geqslant 8$.

Proof. From Theorems 2.1 and 2.2, the graph $S'(K_{1,n})$ is k-product cordial if $n \geqslant \lfloor \frac{k}{2} \rfloor$ and $k \geqslant 4$.
For $k \geqslant 5$, $1 \leqslant n \leqslant \lfloor \frac{k}{3} \rfloor - 1$ if $k \equiv 1, 0 \pmod 3$ and $1 \leqslant n \leqslant \lceil \frac{k}{3} \rceil - 1$ if $k \equiv 2 \pmod 3$.
For k is even and $1 \leqslant i \leqslant n$, we assign $f(u) = 1$, $f(v) = k-1$, $f(u_1) = 0$, $f(u_2) = \frac{k}{2}$ and $f(v_i) = \frac{k}{2} + i$.
For $3 \leqslant i \leqslant n$, we assign $f(u_i) = \begin{cases} \frac{i-1}{2} + 1 & \text{if } i \text{ is odd} \\ k - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$
For k is odd and $1 \leqslant i \leqslant n$, we assign
$f(u) = 1$, $f(v) = k-1$, $f(u_1) = 0$, and $f(v_i) = \lfloor \frac{k}{2} \rfloor + i$.
For $2 \leqslant i \leqslant n$, we assign $f(u_i) = \begin{cases} \frac{i}{2} + 1 & \text{if } i \text{ is even} \\ k - 1 - \frac{i-1}{2} & \text{if } i \text{ is odd.} \end{cases}$
From the above labeling pattern, we have
for $n = 1$ and k is even,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k-1, 0, \frac{k}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$$
for $n = 2$ and k is even,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k-1, 0, \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} + 2 \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geqslant 3$, n and k are even,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 1, k-1, 2, k-2, ..., \frac{n}{2}, k - \frac{n}{2}, \\ & \quad \frac{k}{2} + 1, \frac{k}{2} + 2, ..., \frac{k}{2} + n \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geqslant 3$, n is odd and k is even,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 1, k-1, 2, k-2, ..., k - \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \\ & \quad \frac{k}{2} + 1, \frac{k}{2} + 2, ..., \frac{k}{2} + n \\ 0 & \text{otherwise.} \end{cases}$$
For $n = 1$ and k is even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2} + 1, \frac{k}{2} - 1 \\ 0 & \text{otherwise} \end{cases}$$
for $n = 2$ and k is even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, \frac{k}{2} + 2, \frac{k}{2} - 2 \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geqslant 3$, n and k are even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k-2, ..., \frac{n}{2}, k - \frac{n}{2}, \\ & \quad \frac{k}{2} + 1, \frac{k}{2} - 1, \frac{k}{2} + 2, \frac{k}{2} - 2, ..., \frac{k}{2} + n, \frac{k}{2} - n \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geqslant 3$, n is odd and k is even,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k-2, ..., k - \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \\ & \frac{k}{2}+1, \frac{k}{2}-1, \frac{k}{2}+2, \frac{k}{2}-2, ..., \frac{k}{2}+n, \frac{k}{2}-n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$ and k is odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k-1, 0, \lfloor \frac{k}{2} \rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geqslant 2$, n and k are odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, ..., \lfloor \frac{n}{2} \rfloor + 1, k - (\lfloor \frac{n}{2} \rfloor + 1), \\ & \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, ..., \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geqslant 2$, n is even and k is odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, ..., k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, ..., \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$ and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geqslant 2$, n and k are odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, ..., \lfloor \frac{n}{2} \rfloor + 1, k - (\lfloor \frac{n}{2} \rfloor + 1), \\ & \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, \lfloor \frac{k}{2} \rfloor + 2, ..., \lfloor \frac{k}{2} \rfloor - (n-1), \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geqslant 2$, n is even and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, ..., k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, \lfloor \frac{k}{2} \rfloor + 2, ..., \lfloor \frac{k}{2} \rfloor - (n-1), \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ for all $0 \leqslant i, j \leqslant k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial if $n \geqslant \lfloor \frac{k}{2} \rfloor$, $1 \leqslant n \leqslant \lfloor \frac{k}{3} \rfloor - 1$ for $k \equiv 1, 0 \pmod 3$ and $1 \leqslant n \leqslant \lceil \frac{k}{3} \rceil - 1$ for $k \equiv 2 \pmod 3$.

Now for $k = 7$ and $n = 2$, we assign $f(u) = 4$, $f(v) = 1$, $f(u_1) = 0$, $f(u_2) = 5$, $f(v_1) = 2$ and $f(v_2) = 3$. From this labeling, we have $v_f(i) = \begin{cases} 0 & \text{if } i = 6 \\ 1 & \text{otherwise} \end{cases}$

and $e_f(i) = \begin{cases} 0 & \text{if } i = 4 \\ 1 & \text{otherwise} \end{cases}$ for $0 \leqslant i \leqslant 6$. Hence, the graph $S'(K_{1,2})$ is 7-product cordial.

Conversely, we assume that the graph $S'(K_{1,n})$ is k-product cordial if $\lfloor \frac{k}{3} \rfloor \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ where $k \equiv 1, 0 \pmod 3$ for $k \geqslant 4$ except 7 and $\lceil \frac{k}{3} \rceil \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ for $k \equiv 2 \pmod 3$ where $k \geqslant 8$. From this hypothesis, we have $v_f(i) = 1 \text{ or } 0$, $e_f(i) = 1 \text{ or } 0$ if $n = \lfloor \frac{k}{3} \rfloor \text{ or } \lceil \frac{k}{3} \rceil$ and $e_f(i) = 1$ or 2 if $\lfloor \frac{k}{3} \rfloor + 1 \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1 \text{ or } \lceil \frac{k}{3} \rceil + 1 \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$. Without loss of generality, we assign 1 and $k-1$ to the vertices u and v respectively. Also, we assign $\{0, 2, 3, ..., k-2\}$ to the remaining vertices u_i and v_i. Now, we have $e_f(1) = e_f(k-1) = 0$, which is a contradiction. Therefore, the graph $S'(K_{1,n})$ is not k-product cordial if $\lfloor \frac{k}{3} \rfloor \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ where $k \equiv 1, 0 \pmod 3$ for $k \geqslant 4$ except 7 and $\lceil \frac{k}{3} \rceil \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ for $k \equiv 2 \pmod 3$ where $k \geqslant 8$. Hence, the graph $S'(K_{1,n})$ is k-product cordial if and only if $n \geqslant 1$ and $k \geqslant 4$ except $\lfloor \frac{k}{3} \rfloor \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 1, 0 \pmod 3$ for $k = 4, 6 \text{ and } k \geqslant 8$ and $\lceil \frac{k}{3} \rceil \leqslant n \leqslant \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 2 \pmod 3$ for $k \geqslant 8$. □

An example of 18-product cordial labeling of $S'(K_{1,5})$ shown in Figure 4.

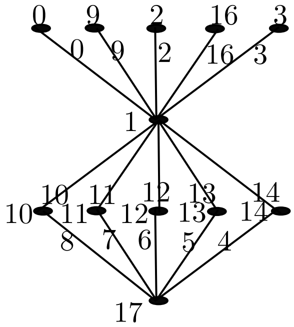

Figure 4: 18-product cordial labeling of $S'(K_{1,5})$

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